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ON THE DISTRIBUTION OF PRODUCTS OF INDEPENDENT  
BETA VARIABLES

by

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On the distribution of products of independent Beta variables.

by

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1. Introduction.

In a number of applications it is necessary to know the properties of the product of random variables; this occurs in particular when the random variables involved have dimensions of a ratio like tolerance expressed in percentages of desired values, fuel consumption per mile, amplification ratios, etc. The special situation discussed in this paper of the product of a number of independent identically distributed random variables arises for instance in the case when some devices designed to amplify a magnitude and having identical characteristics are connected in series. If  $x_i$  is the random variable describing the amplification by the  $i$ th device the total amplification  $X = x_1 x_2 \dots x_n$  is also a random variable and it is important to know the distribution of this product. Examples of engineering applications involving products and quotients of random variables can be found, for instance, in Donahue [4].

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It was shown by Springer and Thompson [9] how to obtain the probability density function (p.d.f.) of products of  $n$  independent, identically distributed random variables by the application of the Mellin transform; they have obtained among others formulae (in the form of rapidly convergent infinite series) for the p.d.f. of the product of  $n$  independent normal variates and  $n$  Cauchy variates and treated also a special case of Beta variates [see below formula (7)]. Their method is a generalization to  $n$  factors of a method presented by Epstein [5] for  $n=2$ . Following Epstein many authors applied the Mellin transform to the study of distribution of products and quotients of random variables; a detailed bibliography can be found in Springer and Thompson [9] and [8]; cf. also Kotlarski [6] and Zolotarev [10].

The Mellin transform of a function  $f(x)$  where  $x > 0$  is defined as

$$(1) \quad M[f(x)/s] = \int_0^{\infty} x^{s-1} f(x) dx .$$

Under certain regularity conditions [c.f. Courant-Hilbert [3] pp. 103-105] this transform considered as a function of complex variable  $s$  admits an inversion integral:

$$(2) \quad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} M[f(x)/s] ds \quad c > 0,$$

where the path of integration is a line parallel to the imaginary axis and to the right of the origin.

An immediate consequence of the definition of the Mellin transform is the following multiplicative property: If  $x$  and  $y$  are independent random variables with the p.d.f.'s  $f(x)$  and  $g(y)$ , respectively and if  $h(z)$  is the p.d.f. of  $z = xy$  then

$$(3) \quad M[h(z)/s] = M[f(x)/s] \cdot M[g(y)/s].$$

Thus to find the p.d.f. of  $y = x_1 x_2 \dots x_n$  where  $x_i$  are independent identically distributed random variables with p.d.f.  $f(x)$  it is sufficient to find the Mellin transform of  $f(x)$ , to take its  $n$ th power and to find the inverse with the aid of formula (2).

The following property of the Mellin transform will be needed later in the discussion of the probability distribution function of the product:

If  $f(x)$  is the p.d.f. of the random variable  $x$  which has finite second moment and if

$$F(x) = \int_0^x f(t) dt, \quad \bar{F}(x) = \int_x^\infty f(t) dt = 1 - F(x),$$

then the Mellin transform of  $F(x)$  is equal to

$$(4) \quad M[F(x)/s] = \int_0^{\infty} F(x) x^{s-1} dx = s^{-1} x^s F(x) \Big|_0^{\infty} + s^{-1} \int_0^{\infty} x^s f(x) dx = s^{-1} M[f(x)/s+1],$$

the vanishing of the term  $s^{-1} x^s F(x)$  when  $x \rightarrow \infty$  following from the existence of the second moment.

It has been shown by the author of this report [7] that the results obtained by Springer and Thompson in [9] for normal random variables can be presented in a somewhat simpler form and that a general formula for any number of factors can be given (still rather unwieldy in the case of large values of  $n$ ). It has been also shown that by a direct application of the Mellin transform similar infinite-series expansions can be derived for the corresponding probability distribution functions; this is useful since it is not always easy to obtain them by the straight-forward integration of the relevant infinite series representing probability densities. Finally, attention has been drawn to the fact implicit in the Springer-Thompson treatment that it is sufficient to evaluate the formulae for the p.d.f. of the product of independent exponentially distributed random variables and from such tables the p.d.f. for products of gamma, normal and Weibull random variables can be quickly evaluated by simple transformations.

The practical usefulness of the results described above is limited by the fact that all the corresponding distributions have infinite ranges while the physical devices to which our mathematical models are to be applied often have finite characteristics. Thus, the problem arises to what extent the methods applied in the above-quoted papers can be modified to obtain similar results for random variables having finite ranges and the first family of random variables which suggests itself and seems to be tractable in this way is the family of Beta distributions with parameters  $p, q$  i.e. the random variables having p.d.f.:

$$(5) \quad \beta(x; p, q) = B(p, q)^{-1} x^{p-1} (1-x)^{q-1} \quad 0 \leq x \leq 1, \quad p \geq 1, \quad q \geq 1,$$

where  $B(p, q)$  is the Euler integral of the first kind:

$$(6) \quad B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx = \Gamma(p) \Gamma(q) / \Gamma(p+q).$$

Notice that in (5) both  $p$  and  $q$  are assumed to be not less than unity; for  $p$  or  $q$  less than 1 the p.d.f. would exhibit an infinite jump at  $x=0$  or  $x=1$  and the inversion formula (2) would not be applicable.

This is not a serious limitation since those "U" and "J" -shaped distributions are of negligible practical importance.

Springer and Thompson [9] gave an explicit formula for the product of  $n$  independent Beta random variables of a special class i.e. of Beta variables with parameters  $p = \alpha + 1$ ,  $q = 1$  defined by the p.d.f.

$$(7) \quad j(x; \alpha+1, 1) = (\alpha+1) x^\alpha \quad 0 \leq x \leq 1, \alpha \geq 0,$$

which they called "monomial" distributions; the rectangular distribution is a member of this class for  $\alpha = 0$ .

In this particular case the application of the Mellin transform gives an immediate answer. Thus

$$(8) \quad M \left[ j(x; \alpha+1, 1) / s \right] = (\alpha+1) \int_0^1 x^{s+\alpha-1} dx = (\alpha+1)(s+\alpha)^{-1}.$$

The Mellin transform of the p.d.f. of the product of  $n$  independent random variables of that kind is equal to

$$(9) \quad M \left[ j_n(x; \alpha+1, 1) / s \right] = (\alpha+1)^n (s+\alpha)^{-n}.$$

But the inverse Mellin transform of  $(s+\alpha)^{-n}$  can be found in tables [see e.g. Bateman Manuscript [2], formula 7.1 (16)] so that

$$(10) \quad j_n(x; \alpha+1, 1) = \begin{cases} (\alpha+1)^n x^\alpha [-\log x]^{n-1} / \Gamma(n) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

There is a wide-spread belief among working statisticians that the problem of the distribution of products of independent random variables is not of a very great interest since "one can always find the distribution of  $y = \log x$ , apply the theorems on the addition of independent random variables  $y_i = \log x_i$  and then revert to the product of the  $x$ 's." It happens so that in the above particular case this procedure works and the application of the Mellin transform (although very attractive) is not essential. Indeed, if we put  $y = \log x$  then it follows from (7) that the distribution of  $y$  given by the p.d.f.  $f(y)$  is the negative exponential distribution with parameter  $\lambda = \alpha + 1$ :

$$f(y) = (\alpha + 1) e^{-y(\alpha + 1)}$$

and it is well known that the sum of  $n$  independent random variables of this kind is a gamma distribution with the shape parameter  $n$  and the scale parameter  $\alpha + 1$  so that

$$f_n(y) = (\alpha + 1)^n e^{-y(\alpha + 1)} y^{n-1} [\Gamma(n)]^{-1}, \quad 0 < y < \infty.$$

Putting  $x = \exp(-y)$  we obtain

$$f_n(x; \alpha + 1, 1) = (\alpha + 1)^n x^\alpha (-\log x)^{n-1} [\Gamma(n)]^{-1},$$

which agrees with (10). However, this seems to be an exceptionally simple situation and in more general cases the passage through the distribution of the sum of logarithms does not make the problem any easier.



## 2. Probability density function of the product of Beta variables.

### 2.1 Simplification of the problem

The Mellin transform of (5) is equal to

$$(11) \quad M\left[\beta(x; p, q) / s\right] = \int_0^1 x^{p+s-2} (1-x)^{q-1} dx / B(p, q) = B(p+s-1, q) / B(p, q) = \\ = \Gamma(p+s-1) \Gamma(p+q) / [\Gamma(p+q+s-1) \Gamma(p)].$$

Denoting by  $\beta_n(x; p, q)$  the p.d.f. of the product of  $n$  such independent random variables we have, in view of (3):

$$(12) \quad M\left[\beta_n(x; p, q) / s\right] = \Gamma^n(p+s-1) \Gamma^n(p+q) / [\Gamma^n(p+q+s-1) \Gamma^n(p)]$$

and (2) yields:

$$\beta_n(x; p, q) = \Gamma^n(p) \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \{x^{-t} \Gamma^n(p+t-1) / \Gamma^n(p+q+t-1)\} dt, \quad c > 0.$$

On substitution  $t = s+n-1$  this becomes:

$$(13) \quad \beta_n(x; p, q) = \Gamma^n(p) \cdot \frac{x^{p-1}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \{x^{-t} \Gamma^n(t) / \Gamma^n(t+q)\} dt, \quad c > 0,$$

the path of integration being still a line parallel to the imaginary axis and to the right of it since  $p > 1$ . But the above formula can be rewritten as

$$\beta_n(x; p, q) = \frac{\Gamma^n(p+q)}{\Gamma^n(p) \Gamma^n(q+1)} \cdot x^{p-1} \frac{\Gamma^n(q+1)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \{x^{-t} \Gamma^n(t) / \Gamma^n(t+q)\} dt, \quad c > 0$$

which, in view of (13) with  $p = 1$  gives

$$(14) \quad \beta_n(x; p, q) = \frac{\Gamma(p+q)}{\Gamma(p) \Gamma(q+1)} x^{p-1} \beta_n(x; 1, q).$$

This shows that it is sufficient to evaluate the p.d.f. of the product of  $n$  random variables of a family of one-parameter Beta variables having  $p = 1$  and arbitrary  $q \geq 1$ ; for  $p > 1$  the relevant p.d.f. are obtained by the above simple transformation. In the subsequent two subsections we shall discuss the derivation of the p.d.f.  $f_n(x; 1, q)$  separately for the case when  $q$  is an integer and the case when it is not an integer.

## 2.2 The case of an integer $q$ .

From (12) we have

$$(15) \quad M[f_n(x; 1, q) | s] = \Gamma^n(s) \Gamma^n(q+1) / \Gamma^n(q+s).$$

But with an integer  $q$ ,

$$\Gamma(q+s) = (q+s-1)(q+s-2) \dots s \cdot \Gamma(s)$$

so that

$$(16) \quad M[f_n(x; 1, q) | s] = \Gamma^n(q+1) \prod_{k=0}^{q-1} (s+k)^{-n},$$

and the inverse formula yields

$$(17) \quad f_n(x; 1, q) = \Gamma^n(q+1) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \prod_{k=0}^{q-1} (s+k)^{-n} ds, \quad c > 0.$$

The integrand in (17) has  $q$  poles of the  $n$ th order for  $s = -j$  ( $j = 0, 1, \dots, q-1$ ) and the integral can be evaluated by contour integration yielding

$$(18) \quad \beta_n(x; 1, q) = \begin{cases} r^{n(q+1)} \sum_{j=0}^{q-1} R(x; n, j) & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x > 1, \end{cases}$$

where

$$(19) \quad R(x; n, j) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left\{ x^{-s} (s+j)^n \prod_{k=0}^{q-1} (s+k)^{-n} \right\}_{s=-j}$$

is the residue of the integrand of (17) at the  $n$ th-order pole at  $s = -j$ . [For the details of contour integration see Appx, para (a), (b), (c)].

Let us write  $\prod_{k=0}^{q-1} \alpha_k$  for the product of  $(q-1)$  factors  $\prod_{k=0}^{q-1} \alpha_k / \alpha_j$

(i.e the product in which the  $j$ th factor has been omitted) and similarly

$\sum_{k=0}^{q-1} \alpha_k$  for the sum in which the  $j$ th term has been omitted. Denoting by  $G_j(s)$  the function in the brackets in (19) we have

$$(20) \quad G_j(s) = x^{-s} \prod_{k=0}^{q-1} (s+k)^{-n}.$$

Before attempting to give a formula for arbitrary integer  $q$  let us discuss two simple cases of  $q = 1$  and  $q = 2$ . For  $q = 1$  we have to consider only one pole and its residue. Here (20) reduces to  $G_0(s) = x^{-s}$ , and (19) gives

$$R(x; n, 0) = \frac{1}{(n-1)!} x^{-s} (-\log x)^{n-1} \Big|_{s=0} = \frac{1}{(n-1)!} (-\log x)^{n-1},$$

so that

$$(21) \quad \beta_n(x; 1, 1) = \begin{cases} \frac{1}{(n-1)!} (-\log x)^{n-1} & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x \geq 1. \end{cases}$$

From (14)

$$(22) \quad \beta_n(x; \rho, 1) = \begin{cases} \rho^n x^{\rho-1} \frac{1}{(n-1)!} (-\log x)^{n-1} & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x \geq 1, \end{cases}$$

which agrees with (10) when  $\rho = \alpha + 1$ .

For  $q = 2$  we have to consider two residues for  $j = 0$  and  $j = 1$  at the poles  $s = 0$  and  $s = -1$ . Formula (20) yields

$$G_0(s) = x^{-s} (s+1)^{-n}, \quad G_1(s) = x^{-s} s^{-n},$$

while formula (19) gives

$$R(x; n, 0) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \{ x^{-s} (s+1)^{-n} \} \Big|_{s=0},$$

$$R(x; n, 1) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \{ x^{-s} s^{-n} \} \Big|_{s=-1}.$$

Applying the Leibniz formula for the  $(n-1)$ th derivative of the product we find

$$\begin{aligned} R(x; n, 0) &= \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} x^{-s} (-\log x)^k (-1)^{n-1-k} n(n+1) \dots (2n-k-2) (s+1)^{-2n+k+1} \Big|_{s=0} = \\ &= \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} (2n-k-2)! (-\log x)^k (-1)^{n-1-k} \end{aligned}$$

and 
$$R(x; n, 1) = \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} x^{-1} (-\log x)^k (-1)^{n-1-k} n(n+1)\dots(2n-k-2) s^{-2n+k+1}, s=-1$$

$$= \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} x^{-1} (-\log x)^k (-1)^n.$$

Hence

$$(23) \quad \mathcal{J}_n(x; -1) = \begin{cases} \sum_{k=0}^{n-1} \binom{n-1}{k} x^{-1} (-\log x)^k \left[ x + (-1)^{k+1} \right], & 0 < x < 1, \\ 0 & \text{for } x \geq 1. \end{cases}$$

Thus, in the two cases  $\gamma = 1$  and  $\gamma = 2$ , an explicit formula for  $\mathcal{J}_n(x; \gamma)$  has been given. For higher values of  $\gamma$  the application of the Leibniz rule would lead to formulae which would become more and more complicated and it is suggested to take advantage of the fact that the logarithmic derivatives of (20) can be easily obtained; once they are found the problem is solved since it is known how to obtain the  $n$ th derivative of a function from its logarithmic derivatives. Indeed, if

$$(24) \quad A(s) = \frac{d}{ds} [\log \phi(s)], \quad \text{and} \quad A^{(r)}(s) = \frac{d^r}{ds^r} A(s),$$

then

$$(25) \quad \mathcal{J}_n(x) = \phi(s) \cdot Z_n[A(s), A'(s), \dots, A^{(n-1)}(s)],$$

where

$$(26) \quad Z_n[A, A', \dots, A^{(n-1)}] = \sum_{k_1, k_2, \dots, k_{n-1}} \frac{n!}{k_1! k_2! \dots k_{n-1}!} A^{(k_1)} A^{(k_2)} \dots A^{(k_{n-1})},$$

here the sum is extended to all the partitions of number  $n$  such that  $n = n_1 k_1 + n_2 k_2 + \dots + n_r k_r$ ;  $A^{(0)}$  denotes  $A(s)$  and  $A^{(r)}$  the  $r$ th derivative of  $A(s)$  with respect to  $s$ .

Clearly,

$$(26a) \quad Z_1 = A,$$

$$(26b) \quad Z_2 = A^2 + A',$$

$$(26c) \quad Z_3 = A^3 + 3AA' + A'',$$

and further formulae can be derived recursively by applying

$$(27) \quad Z_{n+1} = AZ_n + Z_n'$$

(Cf. for example [7]).

In our case  $G_j(s)$  is given by (20) and

$$(28) \quad \log G_j(s) = -s \log x - n \sum_{k=0}^{j-1} \log(s+k).$$

The logarithmic derivative of (20) is

$$A_j(s) = -\log x - n \sum_{k=0}^{q-1} \nu_j (s+k)^{-1},$$

and

$$A_j^{(r)}(s) = (-1)^{r-1} n \nu_j \sum_{k=0}^{q-1} (s+k)^{-(r+1)} \quad (r=1, 2, \dots).$$

Since, according to (19) these derivatives are to be taken for  $s = -j$

we have

$$\begin{aligned} G_j(-j) &= x^j \prod_{k=0}^{q-1} (k-j)^{-\nu_j}, \\ (29) \quad A_j(-j) &= -\log x - n \sum_{k=0}^{q-1} \nu_j (k-j)^{-1}, \\ A_j^{(r)}(-j) &= (-1)^{r-1} n \nu_j \sum_{k=0}^{q-1} (k-j)^{-(r+1)}, \end{aligned}$$

and the final formula for the p.d.f. of the product is:

$$(30) \quad \beta_n(x; 1, q) = \frac{1}{(n-1)!} r^n(q+1) \sum_{j=0}^{q-1} x^j \prod_{k=0}^{q-1} (k-j)^{-\nu_j} Z_{n-1} [A_j, A_j', \dots, A_j^{(n-1)}],$$

where the arguments of the  $Z_{n-1}$ -function are given by (29).

Example. In the case of a rectangular distribution  $p = q = 1$ ,

$$G_0(s) = x^{-s}, \quad \log G_0(s) = -s \log x, \quad A_0(s) = -\log x,$$

all the derivatives of  $A_0(s)$  with respect to  $s$  vanishing. Hence  $Z_{n-1} = A_0^{n-1}$

and  $\beta_n(x; 1, 1) = \frac{1}{(n-1)!} (-\log x)^{n-1}$ , which agrees with (21).

Example. In the case  $p = 1, q = 2$  it is easy to verify that

$$G_0(s) = x^{-s}(s+1)^{-n}, \quad A_0(s) = -\log x - n(s+1)^{-1}, \quad A_0^{(r)}(s) = n r! (s+1)^{-(r+1)} (-1)^{r+1} \\ G_1(s) = x^{-s} s^{-n}, \quad A_1(s) = -\log x - n s^{-1}, \quad A_1^{(r)}(s) = n r! s^{-(r+1)} (-1)^{r+1}$$

Substituting  $s = 0$  in the first line,  $s = -1$  in the second we have

$$G_0(0) = 1, \quad A_0(0) = -\log x - n, \quad A_0^{(r)}(0) = n r! (-1)^{r+1} \\ G_1(-1) = x(-1)^{-n}, \quad A_1(-1) = -\log x + n, \quad A_1^{(r)}(-1) = n r!$$

Formula (30) and (26a) gives for  $n = 2$ :

$$(31) \quad \beta_2(x; 1, 2) = 4 \left\{ (-\log x - 2) + x(-\log x + 2) \right\},$$

and formula (30) and (26b) gives for  $n = 3$ :

$$(32) \quad \beta_3(x; 1, 2) = 4 \left\{ (-\log x - 3)^2 + 3 - x[(-\log x + 3)^2 + 3] \right\};$$

these results can be verified by putting  $n = 2, n = 3$  etc. in (23).

### 2.3 The case when $q$ is not an integer.

From (12)

$$(33) \quad M[\beta_n(x; 1, q)/s] = \Gamma^n(s) \Gamma^n(q+1) / \Gamma^n(q+s+1)$$

and

$$(34) \quad \beta_n(x; 1, q) = \Gamma^n(q+1) \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{\Gamma(s)}{\Gamma(q+s+1)} x^{-s} ds, \quad C > 0.$$

Here the integrand has an infinity of poles of the  $n$ th-order at  $s = -j$  ( $j = 0, 1, \dots$ ) and the integral in (34) can be evaluated by contour integration yielding



$$(35) \quad \zeta_n(x; 1, q) = \begin{cases} \Gamma^n(q+1) \sum_{j=0}^{\infty} R(x; n, j) & \text{for } 0 < x < 1, \\ 0 & \text{for } x \gg 1, \end{cases}$$

where

$$(36) \quad R(x; n, j) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left\{ x^{-s} \frac{\Gamma^n(s)}{\Gamma^n(q+s)} (s+j)^n \right\}_{s=-j}$$

is the residue of the integrand of (34) at the  $n$ th-order pole at  $s = -j$ .

(For the details of contour integration see Appx, paras (d) - (j)). Clearly

$$\Gamma(s)(s+j) = \frac{\Gamma(s+j+1)}{(s+j-1)(s+j-2)\dots s} = \Gamma(s+j+1) \prod_{k=0}^{j-1} (s+k)^{-1} \quad (j=1, 2, \dots)$$

and for  $j=0$  the above identity is still valid if we agree to interpret the "empty" product  $\prod_{k=0}^{-1}$  as unity. Hence

$$(37) \quad R(x; n, j) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left\{ x^{-s} \Gamma^n(s+j+1) \Gamma^{-n}(q+s) \prod_{k=0}^{j-1} (s+k)^{-n} \right\}_{s=-j}.$$

Denoting by  $G_j(s)$  the function in the brackets in (37)

we have

$$G_j(s) = x^{-s} \Gamma^n(s+j+1) \Gamma^{-n}(q+s) \prod_{k=0}^{j-1} (s+k)^{-n},$$

$$\log G_j(s) = -s \log x + n \log \Gamma(s+j+1) - n \log \Gamma(q+s) - n \sum_{k=0}^{j-1} \log(s+k)$$

(interpreting again the "empty" sum  $\sum_{k=0}^{-1}$  as zero to retain the validity of the above for  $j=0$ .)

The successive logarithmic derivatives of  $G_j(s)$  are

$$\begin{aligned}
 A_j(s) &= -\log x + n\psi(s+j+1) - n\psi(q+s) - n \sum_{k=0}^{j-1} (s+k)^{-1} \\
 A_j'(s) &= n\psi'(s+j+1) - n\psi'(q+s) + n \sum_{k=0}^{j-1} (s+k)^{-2} \\
 (38) \quad A_j^{(r)}(s) &= n\psi^{(r)}(s+j+1) - n\psi^{(r)}(q+s) + (-1)^{r-1} r! \sum_{k=0}^{j-1} (s+k)^{-(r+1)},
 \end{aligned}$$

where  $\psi(\cdot)$  denotes the logarithmic derivative of  $\Gamma(\cdot)$  (the Euler "Psi"-function) and  $\psi^{(r)}(\cdot)$  its successive derivatives.

Consequently

$$\begin{aligned}
 G_j(-j) &= x^j \prod_{k=0}^{j-1} (k-j)^{-n} / \Gamma^n(q-j) = \frac{x^j (-1)^{jn}}{(j!)^n \Gamma^n(q-j)}, \\
 (39) \quad A_j(-j) &= -\log x + n\psi(1) - n\psi(q-j) + n \sum_{k=1}^j k^{-1}, \\
 A_j^{(r)}(-j) &= n\psi^{(r)}(1) - n\psi^{(r)}(q-j) + n r! \sum_{k=1}^j k^{-(r+1)},
 \end{aligned}$$

and by the argument which has been applied in derivation of formula (30)

we have

$$(40) \quad \beta_n(x; 1, \gamma) = \frac{\Gamma^n(q+1)}{(n-1)!} \sum_{j=0}^{\infty} x^j \frac{(-1)^{jn}}{(j!)^n \Gamma^n(q-j)} Z_{n-1}[A_j, A_j', \dots, A_j^{(n-2)}],$$

where the values to be put into  $Z_{n-1}$ -function are given by (39).

The infinite series (40) is absolutely convergent for  $0 < x < 1$ . Indeed, the coefficient  $\Gamma^n(q+1)/(j!)^n \Gamma^n(q-j)$  is equal to the  $n$ th power of  $Q_j = \frac{q(q-1)\dots(q-j)}{1 \cdot 2 \dots j}$  ( $j = 0, 1, \dots$ ). For  $j = 0, 1, \dots, [q]$  there are  $[q] + 1$  positive values of  $Q_j$ ; let  $Q$  be their maximum. For  $j \geq [q] + 1$ ,  $\frac{1}{j} \frac{q-j}{j} = (j-1)/j < 1$  so that  $|Q_j| < Q$ . Further, the values to be put into the

$\zeta$ -function are given by (39) and since  $q$  is not an integer the series expansions of  $\psi^{(n)}(y)$  valid for  $y \neq 0, -1, -2, \dots$  can be applied to evaluate  $\psi^{(n)}(q-j)$  [see e.g. [1] formula (6.4.10)]. Thus  $|\psi^{(n)}(q-j)| \leq r! \sum_{k=0}^{\infty} |q-j+k|^{-r} \leq r! \left\{ (q-[q])^{-r} \sum_{k=0}^{\infty} \frac{1}{(q-j+k)^r} + \sum_{k=1}^{\infty} k^{-r} \right\}$ . This is bounded from above by a constant independent of  $j$  and the same applies to the remaining terms of  $A_j^{(n)}$  [see e.g. [1] formula (6.4.2)]. The terms of  $A_j$  are not bounded, but, by a similar argument it can be shown that they are of the order of  $O(\log j)$  for any fixed  $x > 0$  [see e.g. [1] formula (6.3.2) and the recurrence formula (6.3.6)]. Since the values (39) enter into the  $Z_{n,x}$ -function as the  $n$ th powers at the most the series (40) is dominated by  $\sum x^j (\log j)^n$  multiplied by a constant and this still shows that it is convergent for positive  $x$  smaller than 1.

### 3. Probability distribution function of the product of Beta variables.

#### General Remarks.

3.1 Let the probability distribution function corresponding to the p.d.f. (5) be

$$(41) \quad B_n(x; p, q) = E^{-1}(p, q) \int_0^x t^{p-1} (1-t)^{q-1} dt = \int_0^x \beta(t; p, q) dt.$$

In view of (4) and (11) we have

$$(42) \quad M[1 - B_n(x; p, q) | s] = s^{-1} \Gamma^n(p+s) \Gamma^n(n+q) / [\Gamma^n(n+q+s) \Gamma^n(p)],$$

and the inversion formula (2) yields

$$(43) \quad 1 - B_n(x; p, q) = \frac{\Gamma^n(p+q)}{\Gamma^n(p)} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \{x^{-s} s^{-1} \Gamma^n(p+s) / \Gamma^n(p+q+s)\} ds, \quad c > 0.$$

It is not possible to simplify the problem by reducing the discussion to the one-parameter family of probability distribution functions in the way similar to that carried out in section 2.1 for the p.d.f.'s. We have to investigate the two-parameter family of probability distribution functions (43) and we shall again discuss separately the cases of integer and non-integer

$q$ .

### 3.2 The case of integer $q$ .

Since, for integer  $q$ ,

$$\Gamma(p+q+s) = (p+q+s-1)(p+q+s-2) \dots (p+s) \Gamma(p+s),$$

(43) becomes

$$(44) \quad 1 - B_n(x; p, q) = \frac{\Gamma^n(p+q)}{\Gamma^n(p)} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \{x^{-s} s^{-1} \prod_{k=0}^{q-1} (p+s-k)^{-1}\} ds, \quad c > 0.$$

The integrand has one single pole at  $s = 0$  and  $q$  poles of the  $n$ th order at  $s = -j - p$  ( $j = 0, 1, \dots, q-1$ ). The integral can be evaluated by contour integration by the method similar to that applied in the evaluation of integral (17) [cf. Appx para k]. Since the residue at  $s = 0$  multiplied by  $\Gamma^n(p+q)/\Gamma^n(p)$  is clearly equal to 1 we have

$$(45) \quad B_n(x; p, q) = \begin{cases} -\frac{r^n(p+q)}{r^n(p)} \sum_{j=0}^{q-1} R^*(x; n, j) & \text{for } 0 < x < 1, \\ 1 & \text{for } x > 1, \end{cases}$$

where  $R^*(x; n, j)$  is the residue of the integral appearing in (44) at the  $n^{\text{th}}$  order pole at  $s = -j - p$  ( $j = 0, 1, \dots, q-1$ ):

$$(46) \quad R^*(x; n, j) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left\{ x^{-s} s^{-1} (s+j+p)^n \prod_{k=0}^{q-1} (p+s+k)^{-n} \right\} \Big|_{s=-j-p}.$$

Retaining the notation introduced in formula (20) we can write the function  $G_j^*(s)$  in the brackets of (46) as

$$(47) \quad G_j^*(s) = x^{-s} s^{-1} \prod_{k=0}^{q-1} (p+s+k)^{-n}.$$

Starting again with the discussion of two simple cases we have for  $q = 1$  only one residue at  $j = 0$ . From (47)  $G_0(s) = x^{-s} s^{-1}$  and

$$R^*(x; n, 0) = \frac{1}{(n-1)!} \sum_{r=0}^{n-1} \binom{n-1}{r} x^{-s} (-\log x)^r \frac{(-1)^{n-r-1}}{(n-r-1)!} s^{-(n-r)},$$

so that

$$R^*(x; n, 0) = - \sum_{r=0}^{n-1} \frac{1}{r!} x^p (-\log x)^r p^{-n+r};$$

$$(48) \quad B_n(x; p, 1) = x^p \sum_{r=0}^{n-1} \frac{p^r}{r!} (-\log x)^r.$$

The derivative of the above is equal to

$$B_n(x; p, 1) = x^{p-1} p^n (-\log x)^{n-1} / (n-1)!,$$

which agrees with (22).

For  $q = 2$  we have to consider two residues for  $j = 0$  and  $j = 1$ :

$$R^p(x; n, 0) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left\{ x^{-s} s^{-1} (p+s+1)^{-n} \right\} / s = -p,$$

$$R^p(x; n, 1) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left\{ x^{-s} s^{-1} (p+s)^{-n} \right\} / s = -p-1.$$

Applying the Leibniz Formula for the  $(n-1)$ th derivative we obtain

$$R^p(x; n, 0) = \frac{1}{(n-1)!} \sum_{r=0}^{n-1} \binom{n-1}{r} \sum_{k=0}^r \binom{r}{k} x^{-s} (-\log x)^k (-1)^{r-k} (r-k)! s^{-(r-k+1)} \frac{(-1)^{n-r}}{(n-1)!} n(n+1) \dots (2n-r-2) (p+s+1)^{-(2n-r-1)} / s = -p,$$

$$R^p(x; n, 1) = \frac{1}{(n-1)!} \sum_{r=0}^{n-1} \binom{n-1}{r} \sum_{k=0}^r \binom{r}{k} x^{-s} (-\log x)^k (-1)^{r-k} (r-k)! s^{-(r-k+1)} \frac{(-1)^{n-r}}{(n-1)!} n(n+1) \dots (2n-r-2) (p+s)^{-(2n-r-1)} / s = -p-1.$$

Evaluating the above at  $s = -p$  and  $s = -p-1$  respectively

$$R^p(x; n, 0) = -\frac{1}{(n-1)!} \sum_{r=0}^{n-1} \binom{n-1}{r} \sum_{k=0}^r x^p (-\log x)^k (r-k)! \binom{r}{k} p^{-r+k-1} \frac{(2n-r-2)!}{(n-1)!} (-1)^{n-r-1},$$

$$R^p(x; n, 1) = -\frac{1}{(n-1)!} \sum_{r=0}^{n-1} \binom{n-1}{r} \sum_{k=0}^r x^{p+1} (-\log x)^k (r-k)! \binom{r}{k} (p+1)^{-r+k-1} \frac{(2n-r-2)!}{(n-1)!} (-1)^{n-r-1},$$

so that

$$(49) \quad B_n(x; p, 2) = (-1)^n p^n (p+1)^n \sum_{r=0}^{n-1} \binom{2n-r-2}{n-1} \sum_{k=0}^r \frac{1}{k!} (-\log x)^k \left[ x^p p^{-r+k-1} \frac{(2n-r-2)!}{(n-1)!} + x^{p+1} (p+1)^{-r+k-1} \frac{(2n-r-2)!}{(n-1)!} \right].$$

The derivative of (49) is equal to

$$B'_n(x; p, 2) = (-1)^n p^n (p+1)^n \sum_{r=0}^{n-1} \binom{2n-r-2}{n-1} \frac{1}{r!} (-\log x)^r \left[ x^p + x^{p+1} (-1)^{r+1} \right],$$

and is equal to  $B'_n(x; p, 2)$

in view of (23) and (14).

Again, as in section 2.2, for higher values of  $q$ , the application of the Leibniz rule would lead to complicated expressions and we shall again evaluate the logarithmic derivative of (47). Here

$$\log G_j^*(x) = -s \log x - \log s - n \sum_{k=0}^{q-1} \binom{j}{k} \log(p+s+k)$$

and the logarithmic derivative of (47) is

$$(50) \quad A_j^*(x) = -\log x - s^{-1} - n \sum_{k=0}^{q-1} \binom{j}{k} (p+s+k)^{-1},$$

while the higher derivatives are

$$(51) \quad A_j^{*(r)}(x) = r! (-1)^{r+1} s^{-(r+1)} + n r! (-1)^{r+1} \sum_{k=0}^{q-1} \binom{j}{k} (p+s+k)^{-(r+1)}.$$

Since according to (46) these derivatives are to be taken for  $s=j^{-1}$

we have

$$(52) \quad \begin{aligned} G_j^*(-j-p) &= x^{j+p} (j+p)^{-1} \prod_{k=0}^{q-1} \binom{j}{k} (k-j)^{-n}, \\ A_j^*(-j-p) &= -\log x + \frac{1}{j+p} - n \sum_{k=0}^{q-1} \binom{j}{k} (k-j)^{-1}, \\ A_j^{*(r)}(-j-p) &= r! (j+p)^{-(r+1)} + n r! \sum_{k=0}^{q-1} \binom{j}{k} (k-j)^{-(r+1)}, \end{aligned}$$

and the final formula for the probability distribution function of the product is given by

$$(53) \quad B_n(x; p, q) = \frac{1}{(-j)!} \frac{r^n (p+q)}{r^n (p)} \sum_{j=0}^{q-1} \frac{x^{p+j}}{p+j} \prod_{k=0}^{q-1} \binom{j}{k} (k-j)^{-n} \sum_{n=1}^{\infty} (A_j^*, A_j^{*1}, \dots, A_j^{*(n-1)}).$$

where the arguments of  $\sum_{n=1}^{\infty}$  function are given by (52).

Example. If  $q = 1$ , according to (53),

$$B_n(x; p, 1) = \frac{p^{n-1}}{(n-1)!} x^p Z_{n-1} \left[ (-\log x + p^{-1}), p^{-2}, \dots, (n-2)! p^{-n+1} \right].$$

This formula is, for high values of  $n$ , more complicated than (48) but it can be verified that, for  $n = 2, 3$  etc., it is equivalent to (48).

### 3.3 The case when $q$ is not an integer.

Since  $q$  is not an integer the integral (43) cannot be written in a simpler form as in the case of (44). The integrand of (43) has a single pole at  $s = 0$ , and an infinity of poles of the  $n$ th order at  $s = -p - j$  ( $j = 0, 1, \dots$ ). The integral in (43) can be evaluated by contour integration by a method similar to that applied in the evaluation of the integral of (34). [cf. Appx, para (k)]. Since the residue at  $s = 0$  multiplied by  $\Gamma(n+q)/\Gamma^2(p)$  equals to 1 we have

$$(54) \quad B_n(x; p, q) = \begin{cases} \frac{\Gamma^2(p+q)}{\Gamma^2(p)} \sum_{j=0}^{\infty} K^*(x; n, j) & \text{for } 0 < x < 1, \\ 1 & \text{for } x \gg 1, \end{cases}$$

where

$$(55) \quad K^*(x; n, j) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left\{ x^{-s-1} \Gamma^2(p+s) (p+s+j)^n / \Gamma^2(p+q) \right\} / s = -p-j$$

is the residue of the integrand of (43) at the  $n$ th-order pole at  $s = -p - j$  ( $j = 0, 1, \dots$ ).



Clearly

$$\Gamma(p+s)(p+s+j) = \frac{\Gamma(s+p+j+1)}{(p+s+j-1)(p+s+j-2)\dots(p+s)} = \Gamma(s+p+j+1) \prod_{k=0}^{j-1} (p+s+k)^{-1},$$

where again an 'empty' product  $\prod_{k=0}^{-1}$  should be interpreted as unity. Hence

$$(56) \quad R^*(x; n, j) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left\{ x^{-s} s^{-1} \Gamma^*(s+p+j+1) \prod_{k=0}^{j-1} (p+s+k)^{-1} / \Gamma^*(p+q+s) \right\} / s = -p-j.$$

Denoting by  $G_j^*(s)$  the function in brackets in (56) we obtain:

$$(57) \quad G_j^*(s) = x^{-s} s^{-1} \Gamma^*(s+p+j+1) \prod_{k=0}^{j-1} (p+s+k)^{-1} / \Gamma^*(p+q+s)$$

and

$$\log G_j^*(s) = -s \log x - \log s + n \log \Gamma(s+p+j+1) - n \log \Gamma(p+q+s) - n \sum_{k=0}^{j-1} \log(p+s+k).$$

The successive logarithmic derivatives of  $G_j^*(s)$  are given by

$$(58) \quad \begin{aligned} A_j^*(s) &= -\log x - s^{-1} + n\psi(s+p+j+1) - n\psi(p+q+s) - n \sum_{k=0}^{j-1} (p+s+k)^{-1}, \\ A_j^{*(r)} &= (-1)^{r+1} r! s^{-(r+1)} + n\psi^{(r)}(s+p+j+1) - n\psi^{(r)}(p+q+s) + (-1)^{r+1} r! \sum_{k=0}^{j-1} (p+s+k)^{-(r+1)}, \end{aligned}$$

and since (55) are to be taken at  $s = -p-j$  we have

$$(59) \quad \begin{aligned} G_j^*(-p-j) &= -x^{p+j} (p+j)! (-1)^{n+j} (j!)^{-1} \Gamma^*(q-j), \\ A_j^*(-p-j) &= -\log x + (p+j)^{-1} + n\psi(1) - n\psi(q-j) + n \sum_{k=1}^j k^{-1}, \\ A_j^{(r)}(-p-j) &= (-1)^{r+1} r! (p+j)^{-r-1} + n\psi^{(r)}(1) - n\psi^{(r)}(q-j) + n r! \sum_{k=1}^j k^{-(r+1)}, \end{aligned}$$

and finally

$$(60) \quad B_n(x; p, q) = \begin{cases} \frac{\Gamma^*(p+q)}{\Gamma^*(p)} \frac{1}{(n-1)!} \sum_{j=0}^{\infty} \frac{x^{p+j}}{p+j} \frac{(-1)^{n+j}}{j!} \frac{1}{\Gamma^*(q-j)} Z_{n-1}[A_j^*, A_j^{(1)}, \dots, A_j^{(n-2)}] & 0 < x < 1, \\ 1 & x \geq 1, \end{cases}$$

where the arguments of  $Z_{n-1}$ -function are given by (59).

The convergence of the infinite series (60) follows from a similar argument as that applied in the case of the infinite series (40).

4. Distribution of products of independent random variables having the same Beta distributions with different scale parameters.

The generalization of the above results to the case important in practical applications when  $y = x_1 x_2 \dots x_n$  and the p.d.f. of the  $i$ th factor has the p.d.f.

$$\alpha_i \beta(\alpha_i x; p, q)$$

is immediate. The p.d.f.  $g_n(y)$  of such a product is equal to:

$$(61) \quad g_n(y) = \alpha_1 \alpha_2 \dots \alpha_n \beta_n(\alpha_1 \alpha_2 \dots \alpha_n y; p, q)$$

and, consequently the probability distribution function is equal to:

$$B_n(\alpha_1 \alpha_2 \dots \alpha_n y; p, q).$$

Indeed, since

$$M[f(x)/s] = \alpha^{-s} M[f(x)/s],$$

we have

$$\begin{aligned} M[g_n(y)/s] &= [\alpha_1 \alpha_2 \dots \alpha_n]^{-s+1} \left\{ M[\beta(x; p, q)/s] \right\}^n \\ &= [\alpha_1 \alpha_2 \dots \alpha_n]^{-s+1} M[\beta_n(\cdot; p, q)/s] = \\ &= \alpha_1 \alpha_2 \dots \alpha_n M[\beta_n(\alpha_1 \alpha_2 \dots \alpha_n y; p, q)/s] \\ &= M[\alpha_1 \alpha_2 \dots \alpha_n \beta_n(\alpha_1 \alpha_2 \dots \alpha_n y; p, q)/s], \end{aligned}$$

which completes the proof of (61) in view of the uniqueness property of the Mellin transform (cf. [3], p. 104, Th. 2).

## APPENDIX

(a) The evaluation of the integral (17) by contour integration leading to formula (18) is achieved in the following way:

We define two close contours  $\mathcal{L}_A$  and  $\mathcal{L}'_A$ .

(1A)  $\mathcal{L}_A$  is composed of the vertical segment of the integration path of (17) between  $c-iA$  and  $c+iA$  and of a half-circle  $\mathcal{H}_A$  of radius  $A$  lying to the right of this segment and having this segment as its diameter.

Similarly

(2A)  $\mathcal{L}'_A$  is composed of the same vertical segment and a half-circle  $\mathcal{H}'_A$  of radius  $A$  lying to the left of this segment and having this segment as its diameter. Let also  $A > q$ .

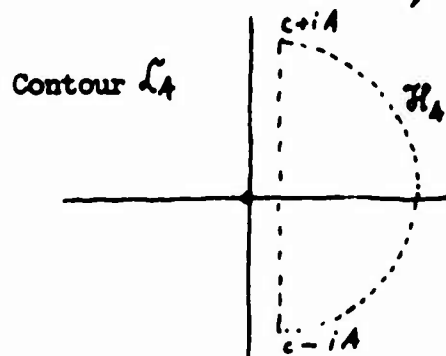


Figure 1.

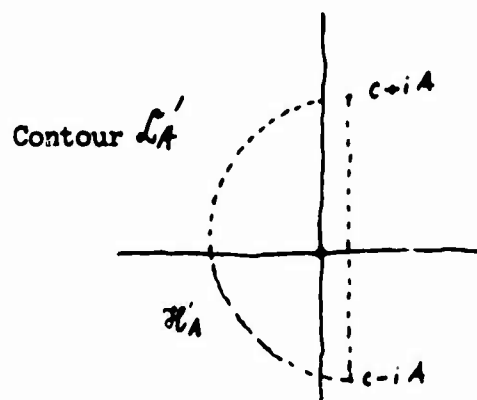


Figure 2.

Since the integrand of (17)

(3A) 
$$f(s) = x^{-s} \prod_{k=0}^{q-1} (s+k)^{-n}$$

is analytic on the contours  $\mathcal{L}_A$  and  $\mathcal{L}'_A$  and its interiors with the exception of  $q$  poles inside the contour  $\mathcal{L}'_A$  we have

$$(4A) \quad \int_{\mathcal{L}_A} f(s) ds = 0, \quad \int_{\mathcal{L}'_A} f(s) ds = \sum_{j=0}^{q-1} R(x; n, j),$$

where  $R(x; n, j)$  is the residue of  $f(s)$  at the pole  $s = -j$  ( $j=0, 1, \dots, q-1$ ).

It remains only to prove that the integral along the half-circle  $\mathcal{H}_A$  tends to zero when  $A$  tends to infinity in the case of  $x > 1$  and that the integral along the half-circle  $\mathcal{H}'_A$  tends to zero when  $A$  tends to infinity in the case of  $0 < x < 1$ .

(b) Let  $x \geq 1$ . Then

$$(5A) \quad \int_{\mathcal{H}_A} f(s) ds = \int_{-\pi/2}^{\pi/2} x^{-(c+A(\cos\phi+Ai\sin\phi))} \prod_{k=0}^{q-1} (c+k+A\cos\phi+Ai\sin\phi)^{-n} A i e^{i\phi} d\phi.$$

But in view of  $\cos\phi$  being positive we have

$$|c+k+A\cos\phi+Ai\sin\phi| = |(c+k)^2 + 2A(c+k)\cos\phi + A^2|^{1/2} > A$$

and the product  $\prod_{k=0}^{q-1}$  in (5A) is dominated by  $A^{-nq}$ . Hence

$$(6A) \quad \left| \int_{\mathcal{H}_A} f(s) ds \right| < 2A^{-nq} \int_0^{\pi/2} x^{-c} x^{-A\cos\phi} A d\phi$$

Since  $\cos\phi > 0$  for  $0 < \phi < \pi/2$  and  $x \geq 1$ ,  $\int_0^{\pi/2} x^{-A\cos\phi} A d\phi \leq A^{\frac{\pi}{2}}$ .

In view of (6A) the integral along the half-circle tends to zero when  $A$  tends to infinity thus completing the proof of the second line of (18).

(c) Let  $0 < x < 1$ . Then

$$(7A) \quad \int_{\mathcal{H}_A'} f(s) ds = \int_{\pi/2}^{3\pi/2} x^{-c-k+A \cos \phi + Ai \sin \phi} \prod_{k=0}^{q-1} (c+k+A \cos \phi + Ai \sin \phi)^{-1} A i e^{i\phi} d\phi$$

But

$$|c+k+A \cos \phi + Ai \sin \phi| = [(c+k)^2 + 2A(c+k) \cos \phi + A^2]^{1/2}$$

and this is greater than

$$|(c+k)^2 - 2A(c+k) + A^2|^{1/2} = |A - c - k| > A - c - q + 1 > A - q$$

since  $c > 0$  could be made less than unity, and it should be borne in

mind that we have assumed  $A > q$ . Hence the product in (7A) is dominated

by the  $(A-q)^{-nq}$  and

$$(8A) \quad \left| \int_{\mathcal{H}_A'} f(s) ds \right| < 2(A-q)^{-nq} x^{-c} \int_{\pi/2}^{\pi} x^{-A \cos \phi} A d\phi.$$

The integral above can be written  $\int_0^{\pi/2} x^{A \cos \phi} A d\phi$  and since  $\cos \phi > 0$

and  $0 < x \leq 1$ , this shows that the left-hand side of (8A) tends to zero

when  $A$  tends to infinity, thus completing the proof of the first line of (18).

(d) The evaluation of the integral (34) by contour integration leading to formula (34) proceeds in a similar way. Notice first that the integrand of (34)

$$(9A) \quad f(s) = x^{-s} \left[ \frac{\Gamma(s)}{\Gamma(s+q)} \right]^n$$

is analytic with the exception of the infinity of poles for  $s = 0, -1, -2, \dots$

Let  $x \geq 1$ . Then on the contour  $\mathcal{C}_A$  as defined above by (1A) we have

$s = u + iv$  with  $u > 0$  and

$$\left| \frac{\Gamma(s)}{\Gamma(s+q)} \right| = \left| \frac{\Gamma(s+1)}{s \Gamma(s+q)} \right| = \left| \frac{1}{s} \right| \cdot \left| \frac{\Gamma(s+1)}{\Gamma(s+q)} \right|.$$

But for complex  $s$  (see [1], formula 6.1.25) and  $s \neq 0, -1, -2, \dots$

$$(10A) \quad |\Gamma(s)| = \Gamma(u) \prod_{r=0}^{\infty} \left[ 1 + \frac{v^2}{(u+r)^2} \right]^{-1}$$

so that

$$(11A) \quad \left| \frac{\Gamma(s+1)}{\Gamma(s+q)} \right| = \frac{\Gamma(u+1)}{\Gamma(u+q)} \prod_{r=0}^{\infty} \frac{1 + \frac{v^2}{(u+q+r)^2}}{1 + \frac{v^2}{(u+1+r)^2}} < 1,$$

since with  $q > 1$  all the factors of the infinite product above are smaller than unity; hence

$$(12A) \quad \left| \frac{\Gamma(s)}{\Gamma(s+q)} \right| < \frac{1}{|s|};$$

by the same argument as that applied to the evaluation of (5A) and (6A) we see that, for  $x \geq 1$ , the integral of (9A) along the half-circle tends to zero when  $A$  tends to infinity thus completing the proof of the second line of (35).

(e) To prove the first line of (35) (for  $0 < x < 1$ ) we define a contour  $\mathcal{L}_A''$  slightly different from that defined by (2A); it is shown in Fig. 3. Let

(15A)  $\mathcal{L}_A''$  be composed of the vertical segment of the integration path of (34) between  $c - Ai$  and  $c + Ai$ , of two horizontal segments joining in the  $s$ -plane points  $-A + Ai$  with  $c + Ai$  and  $-A - Ai$  with  $c - Ai$  and finally of the vertical segment joining point  $-A - Ai$  with  $-A + Ai$ . To avoid the poles of function (9A) we chose a sequence of  $A$ -values as  $A = m + \frac{1}{2}$  and we are going to show that the integral of (9A) along the path composed of segments marked (i) - (v) in Fig. 3 tends to zero when  $A$  tends to infinity assuming the values  $A = m + \frac{1}{2}$ , ( $m = 1, 2, \dots$ ).

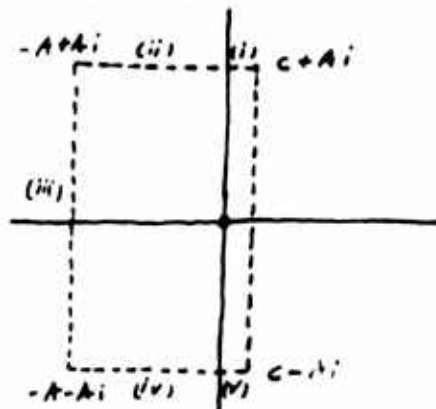


Figure 3

(f) In the integral along (v) the real part of  $s = u - Ai$  is positive and by (12A) the integral is dominated by

$$\left| \int_{(v)} f(s) ds \right| < \left| \int_0^c e^{-(u-Ai)} \frac{du}{(u-Ai)^n} \right| < \int_0^c x^{-u} \frac{du}{A^n} = \frac{1}{A^n} \cdot \frac{x^{-c} - 1}{-\log x}$$

which tends to zero when  $A$  tends to infinity; the same applies to the integral along (i).

(g) To show that the integrals along segments (ii), (iii), (iv), where the real part of  $s$  is negative, have the same property notice that for any non-integer  $z$  (cf. for example [1], formula 5.1.17):

$$(14A) \quad \Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

so that

$$\Gamma(-z+1) = \frac{\pi}{\Gamma(z) \sin(\pi z)}$$

and

$$\Gamma(-z+q) = \frac{\pi}{\Gamma(z-q+1) \sin[\pi(z-q+1)]}.$$

Writing in the above  $-s$  for  $z$  we have

$$\frac{\Gamma(s+1)}{\Gamma(s+q)} = \frac{\Gamma(-s-q+1) \sin[\pi(-s-q+1)]}{\Gamma(-s) \sin(-\pi s)}.$$

But in the integrals which we are going to evaluate the real part of  $-s$  is positive and we can apply (10A) which, in view of  $q > 1$  yields

$$(15A) \quad \left| \frac{\Gamma(s+1)}{\Gamma(s+q)} \right| < \left| \frac{\sin[\pi(-s-q+1)]}{\sin(-\pi s)} \right|.$$

(h) In the integral along (111)  $s = -A+iv$  and in view of (15A)

$$\left| \int_{(ii)} f(s) ds \right| < \left| \int_{-A}^A x^{A-iv} \frac{\Gamma^q(-A+1+iv)}{\Gamma^q(-A+q+iv)} \frac{iv}{(-A+iv)^n} \right| < x^A \int_{-A}^A \left| \frac{\sin[\pi(A-q+1-iv)]}{\sin(\pi(A-iv))} \right| \frac{dv}{(A^2+v^2)^{n/2}}.$$

It is known that if  $z = u+iv$  then  $|\sin z| = [\sin^2 u + \sinh^2 v]^{\frac{1}{2}}$  (cf. for example [1], formula 4.3.59)

so that

$$\left| \int_{(ii)} f(s) ds \right| < \frac{x^A}{A^n} \int_{-A}^A \left[ \frac{\sin^2[\pi(A-q+1)] + \sinh^2 v}{\sin^2(\pi A) + \sinh^2 v} \right]^{\frac{n}{2}} dv.$$



Recalling that  $A = m + \frac{1}{2}$  we have  $\sin^2 \pi A = 1$  and the integrand above is not greater than unity; the integral is dominated by  $2x^4/A^{n-1}$  which tends to zero when  $A$  tends to infinity.

(i) In the integral along (iv)  $s = u - iA$  and

$$\int_{(iv)} f(s) ds = \int_{-A}^0 x^{-u} \frac{r^n(u-iA+1)}{r^n(u-iA+q)} \frac{du}{(u-iA)^n}.$$

In view of (15A) this is dominated by

$$\begin{aligned} & \int_{-A}^0 x^{-u} \left| \frac{\sin[\pi(-u-q+1+iA)]}{\sin[\pi(-u+iA)]} \right| \frac{du}{(u^2+A^2)^{n/2}} < \\ & < \frac{1}{A^n} \int_{-A}^0 x^{-u} \left| \frac{\sin^2 \pi(-u-q+1) + \sinh^2 A}{\sin^2 \pi u + \sinh^2 A} \right|^{n/2} du < \frac{1}{A^n} \int_{-A}^0 x^{-u} \left| \frac{1 + \sinh^2 A}{\sinh^2 A} \right|^{n/2} du \\ & = \frac{1}{A^n} \left[ \frac{\cosh^2 A}{\sinh^2 A} \right]^{n/2} \frac{1-x^4}{-\log x}, \end{aligned}$$

and this tends to zero when  $A$  tends to infinity; the same applies to the integral along (ii).

(j) When  $A = m + \frac{1}{2}$  the contour includes in its interior  $m+1$  poles of function (9A) and with  $m$  tending to infinity the integral (34) is equal to

$$r^n(q+1) \lim_{m \rightarrow \infty} \sum_{j=0}^m R(x; n, j),$$

which is the infinite convergent series (40).

(k) The evaluation of integral (44) by contour integration proceeds in a similar way as that of integral (17) and of integral (43) (in the case of  $q$  not being an integer) in a similar way as that applied in the evaluation of integral (34). In fact the task is easier with the additional factor  $s^{-1}$  appearing in the relevant integrals.

### Summary

The important problem of finding the probability density function of the product of a number of independent identically distributed random variables was investigated by Springer and Thompson [8], [9] who, by the application of the Mellin transform, obtained the answer in the case of the normal and the Cauchy random variables. Further discussion by the author of this report [7] has drawn attention to the fact that the results obtained by them are easily applicable to the cases of exponential, gamma and Weibull distributions and that the probability distribution functions can be also obtained directly by the application of the Mellin transform without integrating the relevant formulae for the probability density functions.

In the present paper it is shown how these methods can be applied to the study of beta random variables; this is of practical importance since most physical devices with which an engineer is dealing have finite characteristics and the results discussed in [8], [9], [7] were mainly concerned with random variables having infinite ranges.

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<p>The problem of finding the probability distribution of a product of a number of identically distributed, independent random variables had been solved by an application of the Mellin transform for normal and Cauchy distributions (Springer and Thompson) and for exponential, Gamma and Weibull distributions (Lomnicki). The present paper shows that it can be solved by similar methods for Beta distributions. This is of practical importance, since most physical quantities with which an engineer is dealing have finite ranges, while all the distributions previously studied had infinite ranges.</p>			

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<p>products of random variables</p> <p>quotients of random variables</p> <p>series-systems</p> <p>system-reliability</p> <p>Mellin transform</p>						